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# Integral transformation of elliptic problems within irregular domains

## Fully developed channel flow

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### Nomenclature

| | | |
|---|---|---|
| $a$ | = | geometric parameter, as defined in Figure 2 |
| $a(t)$ | = | coefficient of $L_t$ operator, equation (6) |
| $b(t)$ | = | coefficient of $L_t$ operator, equation (6) |
| $C$ | = | source team defined in equation (39) |
| $d(x)$ | = | coefficient of $L$ operator, equation (7) |
| $D_h$ | = | hydraulic diameter |
| $f$ | = | friction factor, equation (51) |
| $fk(x)$ | = | source friction in boundary conditions, equations (4)(5) |
| $h$ | = | geometric parameter, as defined in Figure 2 |
| $H$ | = | geometric parameter, ($H$ = 1–$h$) |
| $K(x)$ | = | coefficient of $L$ operator, equation (6) |
| $m$ | = | shape factor, h/a |
| $N$ | = | order of truncated system |
| $N_i(t)$ | = | normalization integral |
| $P(z)$ | = | average pressure at duct cross section, equation (39) |
| $P(x,t)$ | = | source function of equation (1) |
| Re | = | Reynolds number |
| $t$ | = | space co-ordinate |
| $t_0,t_1$ | = | limits of $t$-boundary |
| $T(x,t)$ | = | potential (velocity distribution) |
| $T_{av}$ | = | average velocity |
| $x$ | = | space co-ordinate |
| $x_0(t)$, $x_1(t)$ | = | limits of $x$-boundary |
| $y$ | = | space co-ordinate (applications) |
| z | = | longitudinal co-ordinate |
| *Greek symbols* | | |
| $\alpha_\kappa$ | = | coefficient of boundary condition, equation (9) |
| $\beta_\kappa$ | = | coefficient of boundary condition, equation (9) |
| $\gamma_\kappa$ | = | coefficient of boundary condition, equation (8) |
| $\delta_\kappa$ | = | coefficient of boundary condition, equation (8) |
| $\varepsilon$ | = | relative error estimator, equation (37) |
| $K_i(x,t)$ | = | normalized eigenfunction, equation (15) |
| $\mu$ | = | fluid viscosity, equation (39) |
| $\mu_i(t)$ | | eigenvalues of problem (2) |
| $\phi_\kappa(t)$ | = | source functions in boundary conditions, equations (2)(3) |
| $\psi(\mu_i(t),x)$ | = | eigenfunctions of problem (2) |

**Note:** The symbols defined above are subject to alteration on occasion

### Introduction

The analysis of fully developed flow inside ducts is of major relevance to the design of heat exchange equipment, towards the optimized selection of passages' shapes with minimum friction and enhanced heat transfer performance. The solution of laminar flow within irregularly shaped channels is then crucial to the proper design of compact heat exchangers, as pointed out by Shah and coworkers[1-4]. Various typical configurations were considered in these studies[1-4]

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and friction factors, as well as heat transfer coefficients in fully developed flows, were systematically presented, as obtained from various sources. Different approximate numerical approaches were employed in obtaining such results, and no true benchmarking approach is readily available to identify possible discrepancies among them.

The mathematical formulation for the analysis of fully developed flow within channels belongs to a class of two-dimensional elliptic diffusion-type problems defined in irregularly shaped regions. When seeking an analytical solution to this class of problems, one naturally searches into the classical solution techniques for diffusion problems, such as the integral transform approach[5-7]. In fact, for regular regions, i.e. when the duct boundaries coincide with constant co-ordinate surfaces on the chosen co-ordinate system, exact solutions are readily obtainable through the classical integral transform method[5-7]. Still for regular domains, some extensions on the classes of problems that could be handled were achieved[8-10] by advancing the ideas in the so-called generalized integral transform technique[7] for the solution of a priori non-transformable problems. Later[11-12], these ideas were further extended to handle a class of irregularly shaped domains, in which the irregular boundaries could be described as functions of the other independent variables, for both elliptic and parabolic multidimensional formulations. These specific solutions for channel flow were then applied in the analysis of a few different duct shapes[13-15], and quantities of practical interest were obtained under different geometric arrangements. All these ideas and many other extensions to classical transformable formulations, handled through the same generalized approach, were recently compiled in[7], including the analysis of non-linear problems in heat and fluid flow. Motivated by this success, the present paper is a follow-up to the developments in[11-15], dealing with the integral transform approach as applied in the hybrid numerical-analytical solution of elliptic problems within irregularly shaped domains. First, a more general formulation is considered, which includes the analysis in reference[11] as a special case, and recent developments on the computational implementation are also pointed out, including the automatic global error control feature of this method. Basically, the original partial differential equation is integrally transformed into an infinite system of coupled second order ordinary differential equations, for the transformed potentials in the direction not eliminated through the integral transformation process. On truncation of this infinite system to a finite order, reliable solvers for boundary value problems are utilized, as available in well-tested subroutine packages[16], offering robust schemes for error control. An adaptive procedure is then implemented, which automatically controls the order of the truncated system, until the user-prescribed accuracy requirements are reached, yielding as a by product a global error estimation for the computed solution, owing to the analytic nature of the final expressions. The use of convergence acceleration schemes is also discussed, based on previous developments[7,17-18]. Second, in order to illustrate such aspects and complement the knowledge base on channel flow, a few different configurations are presented here, for which benchmark results were not available or could be incomplete within the available source data. Therefore, three additional geometries were here analysed, namely, the isosceles triangular duct and segmented duct configurations of both circular and elliptical cross-sections. Numerical results for the longitudinal velocity field, as well as for the related friction factors, were obtained, within prescribed accuracy. Also, the convergence rates of the proposed eigenfunction expansions are illustrated.

The present approach is readily extendible to non-linear problems, by letting the source terms in both the original PDE and associated boundary conditions also depend on the potential itself, as demonstrated in different applications involving non-linear formulations[7,19-29] handled through this same approach.

## Diffusion within irregular domains – formal solution

In this section we consider the analytical solution of steady-state diffusion problems in a class of irregularly shaped regions, where the boundary defined by one of the spatial variables can be given as a function of another coordinate[7,11-12]. A reasonably general formulation of a two-dimensional elliptic problem is adopted, and we let $t$ be the space variable that will not be transformed through application of the integral transform process, and $x$ is the space variable that shall be eliminated. Also, let $Lt$ be the linear differential operator associated with the variable $t$ and let the boundary points in the variable $x$ be expressed in terms of the space variable $t$, i.e. $x_0 \equiv x_0(t)$ and/or $x_1 \equiv x_1(t)$. Then, the problem formulation (see Figure 1) is given as:

$$[w(x)L_t + L]\, T(x,t) = P(x,t) \;, \quad \text{in } t_0 < t < t_1 \;, \; x_0(t) < x < x_1(t) \tag{1}$$

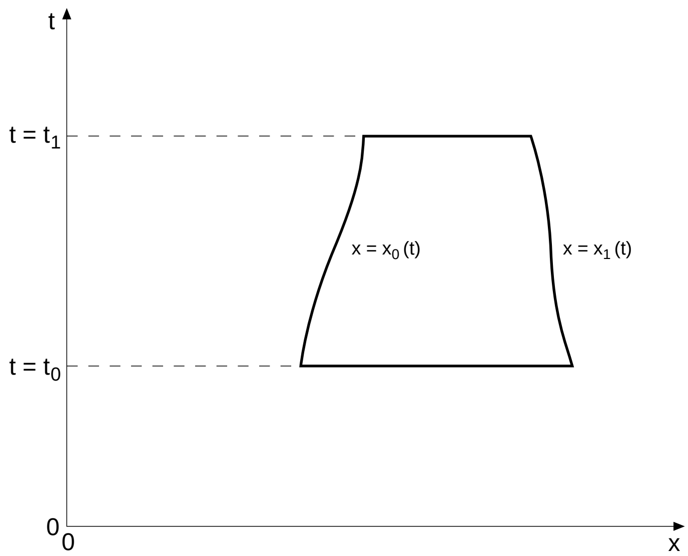


**Figure 1.** Geometry and co-ordinate system for problem (1)

$$B_{x,k}\, T(x_k,t) = \phi_k(t) \quad , \; k = 0,1 \quad , \; t_0 < t < t_1 \tag{2)(3}$$

$$B_{t,k}\, T(x,t) = f_k(x) \quad , \text{ at } \; t = t_k \quad , \; k = 0,1 \tag{4)(5}$$

where the operators in the differential equation are given by:

$$L_t \equiv -a(t) \frac{\partial}{\partial t} [b(t) \frac{\partial}{\partial t}] \tag{6}$$

$$L \equiv -\frac{\partial}{\partial x} [K(x) \frac{\partial}{\partial x}] + d(x) \tag{7}$$

and the boundary condition operators are:

$$B_{t,k} = [\delta_k - (-1)^k \gamma_k \frac{\partial}{\partial t}] \tag{8}$$

$$B_{x,k} = [\alpha_k - (-1)^k \beta_k K(x) \frac{\partial}{\partial x}] \tag{9}$$

The general solution to this problem in regular domains is provided in[5-6], by applying the integral transform technique to remove from system (1) only the partial derivatives associated with the differential operator *L*. The system is then reduced to a decoupled system of second-order, ordinary differential equations in the space variable *t*. If the domain is irregular, i.e. if the chosen co-ordinate system does not coincide with the bounding surfaces, these solutions are not directly applicable. However, the ideas in the generalized integral transform technique can be extended to yield analytical solutions to this class of problems, as now shown.

The appropriate eigenvalue problem is taken as:

$$L\psi(\mu_i(t),x) = \mu_i^2(t) w(x) \psi(\mu_i(t),x) \; , \text{ in } \; x_0(t) < x < x_1(t) \tag{10}$$

with boundary conditions

$$B_{x,k}\, \psi(\mu_i(t),x) = 0 \quad , \text{ at } \; x \equiv x_k(t) \quad , k = 0,1 \tag{11)(12}$$

and the solution of this *t*-dependent eigenvalue problem is assumed to be known at this point.

The integral transform pair, with a symmetric kernel, is then obtained as:

$$\overline{T}_i(t) = \int_{x_0(t)}^{x_1(t)} w(x)\, K_i(x,t)\; T(x,t)\, dx \quad , \quad \text{transform} \tag{13}$$

$$T(x,t) = \sum_{i=1}^{\infty} K_i(x,t)\, \overline{T}_i(t) \quad , \quad \text{inversion} \tag{14}$$

where

$$K_i(x,t) = \frac{\psi(\mu_i(t),x)}{N_i^{1/2}(t)} \tag{15}$$

and the normalization integral is given by:

$$N_i(t) = \int_{x_0(t)}^{x_1(t)} w(x)\,[\psi(\mu_i(t),x)]^2\,dx \tag{16}$$

We now operate on equation (l) by $\int_{x_0(t)}^{x_1(t)} K_i(x,t)$ dx, to obtain:

$$\int_{x_0(t)}^{x_1(t)} w(x)\,K_i(x,t)\,L_t\,T(x,t)\,dx + \mu_i^2(t)\overline{T}_i(t) = \overline{g}_i(t) \ , \tag{17}$$

where,

$$\overline{g}_i(t) = \int_{x_0(t)}^{x_1(t)} K_i(x,t)\ P(x,t)\,dx + \sum_{k=0}^{1} \phi_k(t) \frac{K_i(x_k,t)+(-1)^k K(x_k)\ \dfrac{\partial K_i(x_k,t)}{\partial x}}{\alpha_k + \beta_k} \tag{18}$$

and the only untransformed term in equation (17), after application of the inversion formula (14), is rewritten as:

$$\int_{x_0(t)}^{x_1(t)} w(x)\,K_i(x,t)\,L_t\,T(x,t)\,dx = -a(t) \sum_{j=1}^{\infty} \{[b(t)A_{ij}^*(t) + b(t)B_{ij}^*(t)]\,\overline{T}_j(t) + + [b(t)\,\delta_{ij} + 2b(t)A_{ij}^*(t)]\ \frac{d\overline{T}_j(t)}{dt} + b(t)\delta_{ij} \frac{d^2\overline{T}_j(t)}{dt^2}\} \tag{19}$$

where,

$$A_{ij}^*(t) = \int_{x_0(t)}^{x_1(t)} w(x)\,K_i(x,t)\ \frac{\partial K_j(x,t)}{\partial t}\ dx \tag{20}$$

$$B_{ij}^*(t) = \int_{x_0(t)}^{x_1(t)} w(x)\,K_i(x,t)\ \frac{\partial^2 K_j(x,t)}{\partial t^2}\ dx \tag{21}$$

$$\delta_{ij} = \int_{x_0(t)}^{x_1(t)} w(x)\,K_i(x,t)K_j(x,t)\ dx \ = \begin{cases} 0\ , & i \neq j \\ 1\ , & i = j \end{cases} \tag{22}$$

and dot denotes differentiation with respect to $t$.

The boundary conditions (4)(5) are now transformed through application of the operator $\int_{x_0(t)}^{x_1(t)} w(x)K_i(x,t)dx$, to yield:

$$\delta_k \overline{T}_i(t) - (-1)^k \gamma_k \int_{x_0(t)}^{x_1(t)} w(x) K_i(x,t) \frac{\partial T(x,t)}{\partial t} dx = \overline{f}_{i,k} \text{ , at } t=t_k \text{ , } k=0,1 \quad (23)(24)$$

where,

$$\overline{f}_{i,k} = \int_{x_0(t_k)}^{x_1(t_k)} w(x) K_i(x,t_k) f_k(x,t_k) dx \text{ , } k = 0,1 \quad (25)(26)$$

Again, through substitution of the inversion formula (14), the untransformed term in equations (23)(24), is given by:

$$\int_{x_0(t)}^{x_1(t)} w(x) K_i(x,t) \frac{\partial T(x,t)}{\partial t} dx = \sum_{j=1}^{\infty} \{ A^*_{ij}(t) \overline{T}_j(t) + \delta_{ij} \frac{d\overline{T}_j(t)}{dt} \} \quad (27)$$

The transformed, complete system is finally given in the form that follows:

$$b(t) \frac{d^2\overline{T}_i(t)}{dt^2} + \sum_{j=1}^{\infty} A_{ij}(t) \frac{d\overline{T}_j(t)}{dt} + \sum_{j=1}^{\infty} B_{ij}(t) \overline{T}_j(t) = \overline{h}_i(t) \text{ ,} \quad (28)$$

$$i = 1,2,\ldots, \quad t_0 < t < t_1$$

with boundary conditions

$$\delta_k \overline{T}_i(t_k) - (-1)^k \gamma_k \{ \frac{d\overline{T}_i(t_k)}{dt} + \sum_{j=1}^{\infty} A^*_{ij}(t_k) \overline{T}_j(t_k) \} = \overline{f}_{i,k} \quad , \quad k = 0,1 \quad (29)(30)$$

where,

$$A_{ij}(t) = b(t)\delta_{ij} + 2b(t)A^*_{ij}(t) \text{ ,} \quad (31)$$

$$B_{ij}(t) = b(t)A^*_{ij}(t) + b(t)B^*_{ij}(t) - \frac{\mu_i^2(t)}{a(t)} \delta_{ij} \quad (32)$$

$$\overline{h}_i(t) = - \frac{\overline{g}_i(t)}{a(t)} \quad (33)$$

So far the analysis is formal and exact, yielding the above denumerable system of coupled second-order ordinary differential equations. Therefore, once the quantities $\overline{T}_i(t)$ have been determined, the inversion formula (14) can be utilized to produce the complete solution. For the sake of obtaining numerical results from this a priori formal solution, a finite system is considered instead, after truncation to a sufficiently large order $N$. Quite accurate results can then be obtained by the application of well-established algorithms for systems of ordinary differential equations with boundary conditions at two points[16], providing an interesting alternative to purely numerical approaches.

In addition, an approximate analytical solution is constructed by letting $j = i$ in the summations of the complete system ((28)(33)). This lowest order solution is then readily obtained through solution of the following decoupled system:

$$b(t)\frac{d^2\overline{T}_{i,l}(t)}{dt^2} + A_{ii}(t)\frac{d\overline{T}_{i,l}(t)}{dt} + B_{ii}(t)\overline{T}_{i,l}(t) = \overline{h}_i(t) \quad , \qquad i = 1,2,\ldots, \quad t_0<t<t_1 \tag{34}$$

$$[\alpha_k - (-1)^k\gamma_k A^*_{ii}(t_k)]\overline{T}_{i,l}(t_k) - (-1)^k\gamma_k \frac{d\overline{T}_{i,l}(t_k)}{dt} = \overline{f}_{i,k} \quad , \qquad k = 0,1 \tag{35)(36}$$

This approximate solution is expected to produce sufficiently accurate results in the context of applications, being more or less accurate in a certain range of the parameters that govern the relative magnitudes of the elements in the coefficients of the differential system.

An analytical iteration of the lowest order solution over the complete system (28)-(33) can be employed to account approximately for the effects of non-diagonal elements in matrices $A_{ij}(t)$ and $B_{ij}(t)$, with an expectancy of accuracy improvement over a wider range of the parameters involved[7, 11-14].

These approximate analytical solutions, although not of central interest in the present work, find some use in the realm of applications when fully explicit results are desired without a substantial computational involvement. On the other hand, the hybrid numerical-analytical solution represented by the complete solution of system(28)-(33)), allows for fully error-controlled solutions, since all the intermediate numerical tasks are performed under the user-prescribed accuracy, but the price must be paid on the computational implementation.

For improved convergence behaviour of the eigenfunction expansion proposed, either filtering particular solutions or an integral balance of the original partial differential equation can be employed, as discussed in[7, 17-18]. Both approaches eliminate, or at least alleviate, the difficulties associated with non-homogeneous boundary and equation source terms, by accounting for their contribution to the final solution in an explicit separated form. Either of the two processes involves essentially an analytical pre-treatment of the original problem, without significantly altering the computational procedure to be described in the following section.

**Computational procedure**

A quite straightforward algorithm can be constructed, including the attractive feature of automatically controlling the global error in the final solution at any selected points. To achieve this goal, the semi-analytic nature of this approach is used in conjunction with well-established subroutines libraries with intensively tested accuracy control schemes. The basic steps in computation are as follows:

(1) The auxiliary eigenvalue problem is solved for the eigenvalues and related normalized eigenfunctions, either in analytic form, when applicable, or through the generalized integral transform technique itself[7].

(2) The transformed boundary conditions are computed, either analytically or, in a general purpose procedure, through adaptive numerical integration, such as in subroutines from the IMSL package[16]. Similarly, those coefficients on the transformed ODE system which are not dependent on the transformed potentials can be evaluated a priori, and therefore save some computational effort during the numerical integration of the ODE system.

(3) The truncated ODE system is then numerically solved through the appropriate numerical tools. Boundary value problems can be handled through subroutine DBVPFD[16], which is a more recent implementation of the well-known PASVA3 code, an adaptive finite-difference program for first order non-linear boundary value problems. This subroutine offers an interesting combination of accuracy control, simplicity in use and reliability, with some compromise in speed and memory requirements when compared to dedicated schemes. In either case, a pre-estimate for the truncation order $N$ can be obtained, for instance, through the lowest order solution. Since all the intermediate numerical tasks are accomplished within user-prescribed accuracy, there remains the need of reaching convergence in the eigenfunction expansions and automatically controlling the truncation order $N$, for a certain number of fully converged digits requested in the final solution, at those positions of interest.

The analytic nature of the inversion formulae allows for a direct testing procedure at each specified position within the medium where a solution is desired, and the truncation order $N$ can be gradually increased, to fit the user global error requirements over all the solution domain. The simple tolerance testing formula employed is written as:

$$\varepsilon = \max_{x \in V} \left| \frac{\sum_{i=N^*}^{N} \frac{1}{N_i^{1/2}} \psi_i(\mathbf{x}) \overline{T}_i(t)}{\sum_{i=1}^{N} \frac{1}{N_i^{1/2}} \psi_i(\mathbf{x}) \overline{T}_i(t)} \right| \qquad (37)$$

where $N$ is increased until $\varepsilon$ fits the user requested global error, and then $N$ is changed to assume the value of $N^*$. Significant computational savings are achieved with respect to a plain numerical integration with a fixed size system.

For such elliptic systems (boundary value problems), in which numerical integration is performed at once for all the solution domain, through an iterative procedure, there is no relative gain in repeating computations for a reduced system size, since a more precise solution is already available. Therefore, it is recommended that integration is started with an underestimated value of $N$, and the truncation order can then be gradually increased in fixed steps, $N+\Delta N$, until convergence is achieved in all desired locations. The lower order results already available then serve as excellent initial guesses for the iterative procedure implemented in the boundary value problem solver, providing a faster solution of the higher order ODE system. The adaptive scheme automatically controls the relative error on the final converged solution, and offers in addition a costless error estimator at completion of the integration.

Through this approach, the numerical task is essentially reduced to the solution of an ODE system and, since this is accomplished through the use of widely available and well-documented subroutine packages, the computational implementation becomes quite straightforward without a significant effect on portability. A few representative applications are now considered, handled by the computational procedure just described.

## Applications

*Isosceles triangular duct*

In order to illustrate the application of the hybrid numerical-analytical solutions presented in the previous sections, we consider fully developed laminar flow inside an isosceles triangular duct, as illustrated in Figure 2, with the objective of obtaining benchmark solutions for the velocity field for arbitrary values of the aspect ratio. An exact solution is available in references[2,4] for $\theta = 45°$ which corresponds to the isosceles right angle triangular duct, but analytical explicit solutions are not available for an arbitrary value of the angle $\theta$ (or the aspect ratio, $m = h/a$), although numerical results are available for some cases[2,4]. The mathematical formulation of the problem for the velocity $T(x,y)$ is given by

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = -C \quad \text{in} \quad 0<y<h, \quad 0<x<x_1(y) \qquad (38)$$

where,

$$C = -\frac{1}{\mu}\frac{dP(z)}{dz} \qquad (39)$$

$$x_1(y) = a\left(1 - \frac{y}{h}\right), \text{ and } m=h/a \qquad (40)$$

and the no-slip and symmetry boundary conditions for the velocity $T(x,y)$ are

$$\partial T(0,y)/\partial x = 0, \quad T[x_1(y),y] = 0 \qquad (41)(42)$$

$$T(x,0) = 0, \qquad T(x,h) = 0 \qquad (43)(44)$$

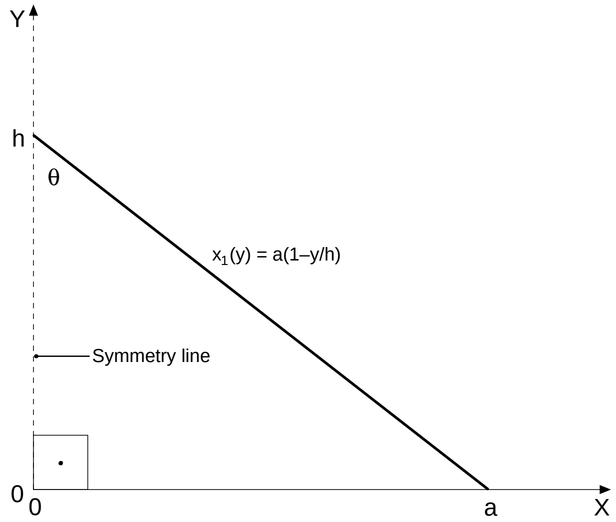


**Figure 2.** Geometry and co-ordinate system for isosceles triangular duct

From direct comparison of the systems (1)-(9) and (38)-(40), the following correspondence between them is obtained

$$a(y) = 1\,,\quad b(y) = 1\,,\quad w(x) = 1\,,\quad K(x) = 1$$
$$d(x) = 0\,,\quad P(x,y) = C\,,\quad f_k(x,t_k) = 0\,,\quad t \rightarrow y$$

and the solution of the corresponding eigenvalue problem in the $x$ variable is readily determined as

$$\psi(\mu_i,x) = \cos(\mu_i x) \tag{45}$$

$$\mu_i(y) = \frac{(2i-1)}{2x_1(y)}\,\pi \tag{46}$$

$$K_i(x,y) = \sqrt{\frac{2}{x_1(y)}}\,\cos\left[\frac{2i-1}{2x_1(y)}\,\pi x\right] \tag{47}$$

The matrix coefficients $A^*_{ij}$ and $B^*_{ij}$ can be readily determined in analytical form, as well as all the other related coefficients. Symbolic manipulation packages can be employed or numerical integrators used to double-check the analytical expressions.

The complete system of transformed equations is given by

$$\frac{d^2\bar{T}_i}{dy^2} + 2\sum_{j=1}^{\infty} A^*_{ij}(y)\,\frac{d\bar{T}_j(y)}{dy} + \sum_{j=1}^{\infty} B_{ij}(y)\bar{T}_j(y) = h^*_i(y)\,,\quad \text{in}\quad 0<y<h\,,\quad i=1,2,3,\ldots \tag{48}$$

subject to the boundary conditions

$$\bar{T}_i(0) = 0 \, , \; \bar{T}_i(\bar{h}) = 0 \quad (49)(50)$$

Equations (48)-(50) can be readily solved numerically and the transforms $\bar{T}_i(t)$ are determined. Once the inversion formula is recalled and the average velocity computed, $T_{av}$, the friction factor is evaluated from the definition

$$f.Re = CD_h^2 / 2T_{av} \quad (51)$$

where $D_h$ is the hydraulic diameter of the channel under consideration.

*Circular segment duct*

In order to illustrate further the application of the hybrid numerical-analytical solutions presented, we consider fully developed laminar flow inside a duct formed by the longitudinal section of a circular tube. Making use of the definitions in Figure 2, with the objective of saving some space, the only required change is in the definition of the irregular boundary, which is now given as:

$$x_1(y) = [1 - (y+H)^2]^{1/2}, \text{ and } H=1-h \quad (52)$$

for a circle of radius unit in dimensionless form, with $H = 0$ for the special case of a hemispherical section.

Again, the required integrals are evaluated analytically, and the computations proceed as in the previous situation.

*Elliptical segment duct*

We have also considered fully developed laminar flow inside a segmented tube of elliptical cross-section. Again, making use of the same definitions in Figure 2, the only required change is in the definition of the irregular boundary, which is now given by:

$$x_1(y) = a\,[1 - y^2]^{1/2}, \quad \text{for } h=1 \quad (53)$$

for an ellipse of semi-axis unit in dimensionless form, with $a = 1$ for the special case of a hemispherical section.

Again, the required integrals are evaluated analytically, and the computations proceed as in the previous situations.

## Results and discussion

Numerical results were obtained for the velocity fields and friction factors in each of the three configurations described above by implementing the proposed computational procedure for the complete solution, system (28)-(31), of the original problem. An accuracy target of $10^{-5}$ was employed throughout the calculations, and the tabulated numerical results are expected to be correct to within ±1 in the fifth significant digit.

Tables I to III illustrate the convergence rates of the eigenfunction expansions for each case considered, respectively, the isosceles triangular duct, the circular segment duct and elliptical segment duct, in terms of the product f.Re, which is essentially an illustration of the convergence of the average velocity, $T_{av}$. Results for different truncation orders of system (28)-(33), namely, $N$ = 4, 8, 12, 16, and 20, are presented, together with the numerical results from[3], for different values of the geometric parameters in each configuration.

| 1/m | $N = 4$ | $N = 8$ | f.Re<br>$N = 12$ | $N = 16$ | Shah[3] |
|---|---|---|---|---|---|
| 0.05 | 12.294 | 12.290 | 12.290 | 12.290 | 12.293 |
| 0.1 | 12.536 | 12.532 | 12.532 | 12.532 | 12.538 |
| 0.2 | 12.900 | 12.896 | 12.896 | 12.896 | 12.904 |
| 0.3 | 13.132 | 13.128 | 13.128 | 13.128 | 13.128 |
| 0.4 | 13.264 | 13.261 | 13.261 | 13.261 | 13.264 |
| 0.5 | 13.325 | 13.322 | 13.322 | 13.322 | 13.322 |
| 0.6 | 13.335 | 13.333 | 13.333 | 13.333 | 13.332 |
| 0.7 | 13.313 | 13.311 | 13.311 | 13.311 | 13.311 |
| 0.8 | 13.270 | 13.268 | 13.268 | 13.268 | 13.275 |
| 0.9 | 13.215 | 13.214 | 13.214 | 13.214 | 13.228 |
| 1.0 | 13.154 | 13.153 | 13.153 | 13.153 | 13.175 |
| 1.2 | 13.027 | 13.027 | 13.027 | 13.027 | 13.076 |
| $3^{-1/2}$ (Equil.) | 13.336 | 13.334 | 13.333 | 13.333 | 13.333 |

**Table I.** Convergence behaviour and reference results for friction factor in the isosceles triangular duct

| H | $N = 4$ | $N = 8$ | f.Re<br>$N = 12$ | $N = 16$ | Shah[3] |
|---|---|---|---|---|---|
| 0.0 | 15.776 | 15.770 | 15.769 | 15.769 | 15.760 |
| 0.1 | 15.763 | 15.758 | 15.757 | 15.757 | 15.747 |
| 0.2 | 15.749 | 15.744 | 15.744 | 15.744 | 15.734 |
| 0.3 | 15.733 | 15.729 | 15.729 | 15.729 | 15.719 |
| 0.4 | 15.716 | 15.712 | 15.712 | 15.712 | 15.703 |
| 0.5 | 15.696 | 15.693 | 15.692 | 15.693 | 15.685 |
| 0.6 | 15.673 | 15.671 | 15.671 | 15.671 | 15.665 |
| 0.7 | 15.649 | 15.647 | 15.647 | 15.647 | 15.643 |
| 0.8 | 15.622 | 15.620 | 15.620 | 15.620 | 15.618 |
| 0.9 | 15.592 | 15.590 | 15.590 | 15.591 | 15.589 |
| 0.95 | 15.575 | 15.574 | 15.574 | 15.575 | 15.572 |

**Table II.** Convergence behaviour and reference results for friction factor in the circular segment duct

**Table III.**
Convergence behaviour and reference results for friction factor in the elliptical segment duct

| a/h | f.Re<br>$N$ = 8 | $N$ = 12 | $N$ = 16 | $N$ = 20 | $N$ = 24 |
|---|---|---|---|---|---|
| 1.0 | 15.767 | 15.766 | 15.766 | 15.766 | 15.766 |
| 1.5 | 16.716 | 16.715 | 16.715 | 16.715 | 16.715 |
| 2.0 | 17.468 | 17.466 | 17.466 | 17.466 | 17.466 |
| 2.5 | 17.982 | 17.980 | 17.980 | 17.980 | 17.980 |
| 3.0 | 18.342 | 18.340 | 18.340 | 18.340 | 18.340 |
| 4.0 | 18.796 | 17.894 | 18.793 | 18.793 | 18.793 |
| 5.0 | 19.059 | 19.057 | 19.057 | 19.056 | 19.056 |
| 6.0 | 19.225 | 19.223 | 19.223 | 19.222 | 19.222 |
| 9.0 | 19.474 | 19.472 | 19.471 | 19.471 | 19.471 |

In all three situations, and for different values of the related parameters, full convergence to five digits is achieved even for $N$ as low as 8, and the columns for the largest $N$ provide a set of benchmark results for reference purposes. With respect to the comparisons against the numerical solutions compiled in[3], a reasonably good agreement was obtained, validating such results up to three or four digits, depending on the range of the geometric parameter for each type of duct. For instance, the results of[3] for the isosceles triangular duct appear to be correct to four digits for the lower values of l/m, becoming less accurate as this factor is increased. The comparison for the elliptical segmented duct is not considered here, since the proposed expression in[3] for this geometry seems to be in error.

Figures 3-5 show isoline plots of the fully converged velocity field inside each duct, in the same respective order, normalized by the average velocity, and for

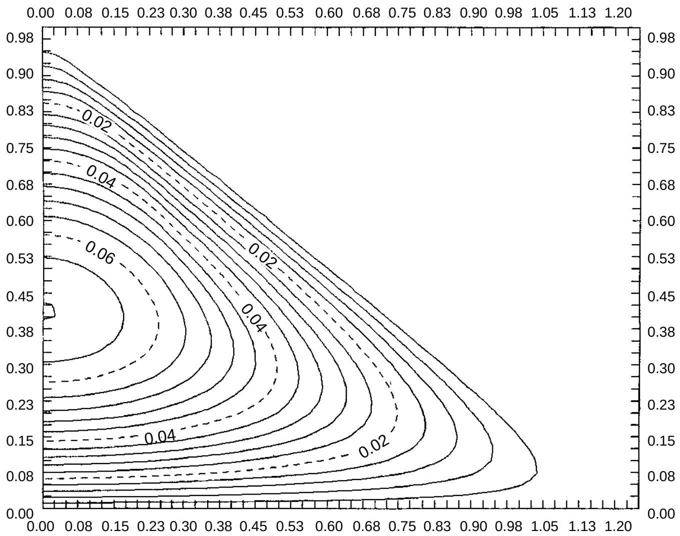


**Figure 3.**
Velocity field inside an isosceles triangular duct (1/m = 1.25)

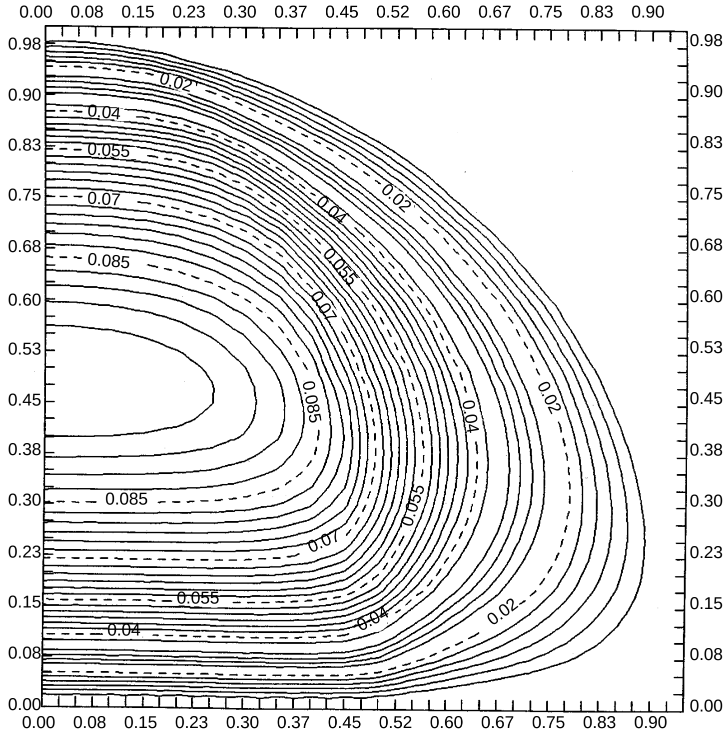


**Figure 4.** Velocity field inside a hemispherical duct ($H$ = 0)

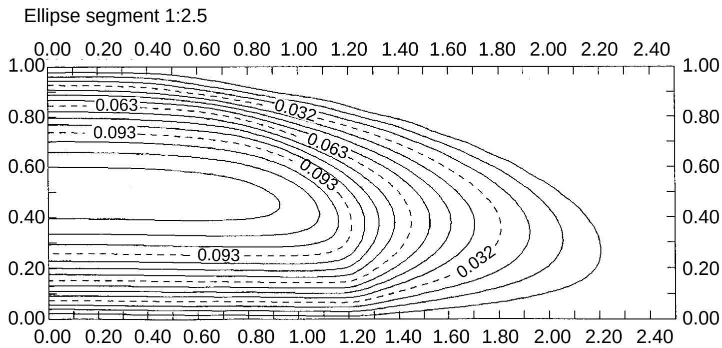


**Figure 5.** Velocity field inside an elliptical segment duct (a/h = 2.5)

some representative values of the governing geometric parameters, namely, 1/m = 1.25 for the isosceles triangular duct, the hemispherical segment duct, and a/h = 2.5 for the elliptical segment duct, which serve to demonstrate the absence of oscillations in the eigenfunction expansions representations of the potential, over the entire solution domain.

The code implemented can be directly employed for the many different geometric configurations typical of compact heat exchangers, as discussed in[2,3], as well as readily extended to more involved situations, such as in turbulent flow applications, non-Newtonian fluids and stratified two-phase flows. The formal solution presented is sufficiently general to include several other applications in heat and fluid flow within irregular domains, either linear or non-linear, provided the irregular boundary can be described as a function of the co-ordinate chosen not to be eliminated through the integral transformation process. For situations when this is initially not feasible, then a domain decomposition needs to precede the application of the ideas here advanced, for each sub-domain.

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